Assignment 11

1. A dyadic rational number is a rational number of the form $k/2^n$ for some integers k and n. Show that all dyadic rational numbers form a dense set in $\mathbb{R}$.

2. Let $\{q_j\}$ be all rational numbers in $[0,1]$. Is it true that $[0,1] \subset \bigcup_j B_{8^{-j}}(q_j)$? Recall that $B_r(q) = (q-r, q+r)$. This gives you an example of a dense, open set.

3. Let $f : X \to Y$ be continuous where X and Y are metric spaces. Let E be dense in X. Prove that $f(E)$ is dense in $f(X)$.

4. Let D be a dense set in the complete metric space X. Show that every uniformly continuous function defined in D can be extended to become a uniformly continuous function in X.

5. Here we present another proof of the separability of $C[0,1]$ without Weierstrass approximation theorem. For each n, divide $[0,1]$ into n many subintervals of length $1/n$ and consider the collection $\mathcal{R}_n$ of all continuous functions which are linear on each subinterval. Furthermore, they must be of the form $bx+a,\ a,b \in \mathbb{Q}$, over each subinterval. Show that $\cup_n \mathbb{R}_n$ forms a countable, dense subset in $C[0,1]$.

6. Show that the boundary of a nonempty open set in a metric space must be closed and nowhere dense. Conversely, every closed, nowhere dense set is the boundary of some open set.

7. Use Baire category theorem to show that transcendental numbers are dense in the set of real numbers.

8. Let $\mathcal{F}$ be a subset of $C(X)$ where X is a complete metric space. Suppose that for each $x \in X$, there exists a constant M depending on x such that $|f(x)| \le M,\ \forall f \in \mathcal{F}$. Prove that there exists an open set G in X and a constant C such that $\sup_{x\in G} |f(x)| \le C$ for all $f \in \mathcal{F}$. Suggestion: Consider the decomposition of X into the sets $X_n = \{x \in X : |f(x)| \le n,\ \ \forall f \in \mathcal{F}\}$.

9. A function is called non-monotonic if if is not monotonic on every subinterval. Show that all non-monotonic functions form a dense set in $C[a,b]$. Hint: Consider the sets
$$\mathcal{E}_n = \{f \in C[a,b] : \exists x \text{ such that } (f(y)-f(x))(y-x) \ge 0,\ \forall y,\ |y-x| \le 1/n\}.$$

10. Optional. A basis of a vector space V is a set consisting of linearly independent vectors satisfying, for each $v \in V$, there exist finitely vectors in this set such that v is a linear combination of these vectors. Show that every basis of a Banach space must be an uncountable set. Recall that a Banach space is a vector space endowed with a norm whose induced metric is a complete one. Hint: Try to decompose the Banach space into union of finite dimensional subspaces. You may assume every finite dimensional subspace of a Banach space is closed. The proof of this fact is not so easy.